SPECTRUM OF PARTIAL INTEGRAL OPERATORS WITH DEGENERATE KERNEL

ESHKABILOV YU.KH., ARZIKULOV G.P., AND HAYDAROV F.H.

Abstract. In the paper we consider self-adjoint partial integral operators of Fredholm type T with a degenerate kernel on the space $L_2([a,b]\times[c,d])$. Essential and discrete spectra of T are described.

1. Introduction

Linear equations and operators involving partial integrals appear in elasticity theory [1, 2, 3], continuum mechanics [2, 4, 5, 6], aerodynamics [7] and in PDE theory [8, 9]. Self-adjoint partial integral operators arise in the theory of Schrödinger operators [10, 11, 12, 13]. Spectrum of a discrete Schrödinger operator H are tightly connected (see [13, 14]) with that of partial integral operators which participate in the presentation of the operator H.

Let Ω_1 and Ω_2 be closed boundary subsets in $\mathbb{R}^{\nu_1}$ and $\mathbb{R}^{\nu_2}$, respectively. Partial integral operator (PIO) of Fredholm type in the space $L_p(\Omega_1\times\Omega_2)$, $p\geq 1$ is an operator of the form [15]:

$$T = T_0 + T_1 + T_2 + K, \tag{1}$$

where operators T_0, T_1, T_2 and K are defined by the following formulas:

$$T_0 f(x,y) = k_0(x,y)f(x,y),$$
$$T_1 f(x,y) = \int_{\Omega_1} k_1(x,s,y)f(s,y)ds,$$
$$T_2 f(x,y) = \int_{\Omega_2} k_2(x,t,y)f(x,t)dt,$$
$$Kf(x,y) = \int_{\Omega_1}\int_{\Omega_2} k(x,y;s,t)f(s,t)dsdt.$$

Here k_0, k_1, k_2 and k are given measurable functions on $\Omega_1\times\Omega_2$, $\Omega_1^2\times\Omega_2^2$, $\Omega_1\times\Omega_2$ and $(\Omega_1\times\Omega_2)^2$, respectively, and all integrals have to be understood in the Lebesgue sense, where $ds = d\mu_1(s), dt = d\mu_2(t)$, $\mu_k(\cdot)$ is the Lebesgue measure on the σ-algebra of subsets Ω_k, $k = 1, 2$.

In 1975, Likhtarnikov and Vitova [16] spectral properties of partial integral operators are studied . In [16], the following restrictions were imposed: $k_1(x,s)\in L_2(\Omega_1\times\Omega_1)$, $k_2(y,t)\in L_2(\Omega_2\times\Omega_2)$ and $T_0 = K = 0$. In [17], spectral properties of PIO with positive kernels were studied (under restriction $T_0 = K = 0$). In Kalitvin and Zabrejko [18] spectral properties of PIO with kernels of two variables

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in $L_p, p \geq 1$ are studied. In [19, 20, 21, 22, 23] for the more general equations with continuous kernels or kernels in $C(L_1)$ spectral properties of the PIO and solvability of partial integral equations in the space $C([a,b] \times [c,d])$ were studied.

Self-adjoint PIO with $T_0 \neq 0$ were first studied in [10], where theorem above essential spectrum is proved. Finiteness and infiniteness of a discrete spectrum of self-adjoint PIO, arising in the theory of Schrödinger operators, are investigated in [11, 12, 14]. In [24] applications of partial integral equations and operators to problems of continuous mechanics, elasticity problems and other problems were considered. Still, some important spectral properties of PIO in the space L_2 are left open. The present paper is dedicated to the mentioned problem for PIO with degenerate kernels from the class L_2.

Let T_1 be a linear integral operator in the space $L_2([a,b] \times [c,d])$ given by the formula

$$(T_1 f)(x,y) = \int_a^b k(x,s,y) f(s,y) ds. \tag{2}$$

Here $k(x,s,y)$ is a measurable function on $[a,b]^2 \times [c,d]$.

The kernel $k(x,s,y)$ of the integral operator T_1 usually satisfies the condition

$$\int_a^b k(x,s,y) f(s,y) ds \in L_2([a,b] \times [c,d]), \quad \forall f \in L_2([a,b] \times [c,d]).$$

Consequently, the operator T_1 is a linear bounded operator on $L_2([a,b] \times [c,d])$. If, in addition, the kernel $k(x,s,y)$ satisfies the condition:

$$k(x,s,y) = \overline{k(s,x,y)}, \quad \text{for almost all } y \in [c,d],$$

then the operator T_1 is a self-adjoint operator on the Hilbert space $L_2([a,b] \times [c,d])$.

Let $\{\varphi_k(x)\}_{k=1}^n$ be a orthonormal system of functions from the $L_2[a,b]$, and let $\{h_k(y)\}_{k=1}^n$ be a system of essential bounded real functions on $[c,d]$.

We define a measurable function $k_1(x,s,y)$ on $[a,b]^2 \times [c,d]$ by the following rule:

$$k_1(x,s,y) = \sum_{k=1}^n \varphi_k(x) \overline{\varphi_k(s)} h_k(y). \tag{3}$$

Then the PIO T_1 with the kernel $k_1(x,s,y)$ is a self-adjoint bounded linear operator on $L_2([a,b] \times [c,d])$.

Let $\{\psi_k(y)\}_{k=\overline{1,m}}$ be a some orthonormal system of functions from the $L_2[c,d]$, $\{p_k(x)\}_{k=\overline{1,m}}$ be a system of essential bounded real functions on $[a,b]$. We define the measurable function $k_2(x,t,y)$ on $[a,b] \times [c,d]^2$ by the following rule:

$$k_2(x,t,y) = \sum_{j=1}^m p_j(x) \psi_j(y) \overline{\psi_j(t)}. \tag{4}$$

Then the PIO T_2 with the kernel $k_2(x,t,y)$:

$$T_2 f(x,y) = \int_c^d k_2(x,t,y) f(x,t) d\mu_2(t)$$

is a linear bounded self-adjoint operator on $L_2([a,b] \times [c,d])$.

For an essential bounded function $\varphi \geq 0$ on the measurable set $\Omega \subset \mathbb{R}^\nu$, we define

$$esssup_\Omega(\varphi) = inf\{C : \mu(\{\xi \in \Omega : \varphi(\xi) > C\}) = 0\},$$

where $\mu(\cdot)$ is the Lebesgue measure on $\mathbb{R}$. For a measurable function φ on the set $\Omega \subset \mathbb{R}^{\nu}$, the number $\lambda \in \mathbb{R}$ is called an essential value of the function φ if

$$\mu(\{\xi \in \Omega : \lambda - \varepsilon < \varphi(\xi) < \lambda + \varepsilon\}) > 0$$

for all $\varepsilon > 0$. We denote by $Essran(\varphi)$ the set of all essential values of the function φ.

In this paper, we study essential and discrete spectra of PIO of the form $T_1 + T_2$ with the degenerate kernels. The resolvent set, spectrum, essential spectrum and discrete spectrum are denoted by ρ, σ, σ_{ess} and σ_{disc},, respectively (see [25]).

2. Spectral property of PIO T_1 and T_2

In this section, we study spectra of PIO T_1 and T_2.

Proposition 2.1. *Zero is an eigenvalue of T_1 of infinite multiplicity. A number $\lambda_0 \neq 0$ is an eigenvalue of T_1 if and only if there exists $1 \leq j_0 \leq n$ such that $\mu_2(h_{j_0}^{-1}(\{\lambda_0\})) > 0$.*

We denote by M the subspace of a Hilbert space $L_2[a,b]$ constructed by the orthogonal system $\{\varphi_1, ..., \varphi_n\}$. Then $\dim M = n$, and for the subspace $\mathcal{H} = L_2[a,b] \ominus M$, we have $\dim \mathcal{H} = \infty$.

Let $\{g_k\}_{k\in\mathbb{N}}$ be an orthonormal basis in $\mathcal{H}$, and $\psi \in L_2[c,d]$, $\|\psi\| = 1$. Then $f_k(x,y) = g_k(x)\psi(y) \in L_2([a,b] \times [c,d])$, $k \in \mathbb{N}$, and the system $\{f_k\}_{k\in\mathbb{N}}$ is orthonormal in $L_2([a,b] \times [c,d])$. Clearly,

$$T_1 f_k = \sum_{i=1}^{n} \int_a^b \varphi_i(x)\overline{\varphi_i(s)} f_k(s,y) d\mu_1(s) = 0, \ \ k \in \mathbb{N},$$

i.e. zero is an eigenvalue of PIO T_1 of infinite multiplicity.
"if" part. Let $\lambda_0 \in \mathbb{C}\backslash\{0\}$ be an eigenvalue of the PIO T_1. Then there exists $f \in L_2([a,b] \times [c,d])$, $\|f\| = 1$ such that

$$T_1 f = \lambda_0 f.$$

We define compact self-adjoint integral operators K_ω in $L_2[a,b]$ as follows:

$$K_\omega \varphi(x) = \int_a^b k_1(x,s,w)\varphi(s) d\mu_1(s), \ \ \omega \in \Omega_0,$$

where

$$\Omega_0 = \{\omega \in [c,d] : p_\omega(x,s) = k_1(x,s,\omega) \in L_2([a,b]^2)\}.$$

We have $\mu_2([c,d] \setminus \Omega_0) = 0$. Put

$$\mathcal{M}_1 = \{\omega \in \Omega_0 : f_\omega(x) = f(x,\omega) \in L_2[a,b]\}.$$

Then $\mu_2(\mathcal{M}_1) > 0$.

We define measurable subsets:

$$\mathcal{D}_k = \{\omega \in [c,d] : h_k(\omega) = \lambda_0\}, \ \ k \in \{1, ..., n\}.$$

Put

$$\mathcal{D}_0 = \bigcup_{k=1}^{n} \mathcal{D}_k.$$

Let $\omega \in \mathcal{M}_1$. Then $K_\omega f_\omega = \lambda_0 f_\omega$, i.e. the number λ_0 is an eigenvalue of operators $K_\omega,\ \omega \in \mathcal{M}_1$. We define by $\left\{\alpha_1^{(\omega)}, ..., \alpha_{n_\omega}^{(\omega)}\right\}$ the set of eigenvalues of the operator K_ω which are different from zero. Then

$$\lambda_0 \in \ \{\alpha_1^{(\omega)}, ..., \alpha_{n_\omega}^{(\omega)}\} \subset \{h_1(\omega), ..., h_n(\omega)\},$$

i.e. there exists $j_0 \in \{1, ..., n\}$ such that $h_{j_0}(\omega) = \lambda_0$.

Consequently, we have $\omega \in \mathcal{D}_0$. Thus, $\mathcal{M}_1 \subset \mathcal{D}_0$. It means that $\mu_2(\mathcal{D}_0) > 0$. Then there exists $j_0 \in \{1, ..., n\}$ such that

$$\mu_2(\{\omega \in [c, d] : h_{j_0}(\omega) = \lambda_0\}) > 0.$$

"only if" part. Let for a $j_0 \in \{1, ..., n\}$ we have $\mu_2(h_{j_0}^{-1}(\{\lambda_0\})) > 0$. Put $\mathcal{D} = h_{j_0}^{-1}(\{\lambda_0\})$. We define the function $\psi \in L_2[c, d]$ in the following way:

$$\psi(y) = \frac{\chi_{\mathcal{D}}(y)}{\sqrt{\mu_2(\mathcal{D})}},\ \ y \in [c, d],$$

where $\chi_G(\cdot)$ is the characteristic function of a set G. Obviously, $\|\psi\| = 1$. Let $f(x, y) = \varphi_{j_0}(x)\psi(y)$. Then $f \in L_2([a, b] \times [c, d])$ and $\|f\| = 1$. On the other hand,

$$T_1 f(x, y) = \sum_{k=1}^{n} \varphi_k(x) \int_a^b \overline{\varphi_k(s)} h_k(y)\varphi_{j_0}(s)\psi(y)d\mu_1(s) = \varphi_{j_0}(x)h_{j_0}(y)\psi(y) = \lambda_0 f(x, y),$$

i.e. the number λ_0 is an eigenvalue of T_1.

We consider the following projectors P_k in the space $L_2([a, b] \times [c, d])$:

$$P_k f(x, y) = \int_a^b \varphi_k(x)\overline{\varphi_k(s)} f(s, y)d\mu_1(s),\ \ k \in \{1, ..., n\}.$$

Let $P = P_1 + ... + P_n$ and $P_0 = E - P$, where E is the identical operator. Then projectors P_i and P_j $(i \neq j)$ are orthogonal.

Proposition 2.2. *If* $\lambda \neq 0$ *and* $\lambda \overline{\in} \bigcup_{k=1}^{n} Essran(h_k)$, *then the operator* $T_1 - \lambda E$ *is invertible in* $L_2([a, b] \times [c, d])$, *and the operator* $(T_1 - \lambda E)^{-1}$ *is bounded in* $L_2([a, b] \times [c, d])$, *moreover*

$$(T_1 - \lambda E)^{-1} f(x, y) = -\frac{1}{\lambda}\left(f(x, y) - \sum_{k=1}^{n} \frac{h_k(y)}{h_k(y) - \lambda} P_k f(x, y)\right).$$

Proof. Let $\lambda \neq 0$ and $\lambda \overline{\in} \bigcup_{k=1}^{n} Essran(h_k)$. We define the operator B_λ on the $L_2([a, b] \times [c, d])$ by the formula:

$$B_\lambda f(x, y) = \sum_{k=1}^{n} \frac{1}{h_k(y) - \lambda} P_k f(x, y) - \frac{1}{\lambda} P_0 f(x, y).$$

It is clear,

$$(T_1 - \lambda E)B_\lambda = B_\lambda(T_1 - \lambda E) = E. \tag{5}$$

For all $\lambda \overline{\in} Essran(h_k) \cup \{0\}$, the operator

$$A_k f(x, y) = \frac{1}{h_k(y) - \lambda} P_k f(x, y),\ \ f \in L_2([a, b] \times [c, d])$$

is a bounded operator in $L_2([a,b]\times[c,d])$. Then the operator B_λ is a bounded operator in $L_2([a,b]\times[c,d])$. For each $\lambda \overline{\in} \{0\}\cup\left(\bigcup\limits_{k=1}^{n} Essran(h_k)\right)$, (5) implies

$$(T-\lambda E)^{-1} = B_\lambda.$$

Hence, we have

$$B_\lambda f(x,y) = -\frac{1}{\lambda}\left(f(x,y) - \sum_{k=1}^{n}\frac{h_k(y)}{h_k(y)-\lambda}P_k f(x,y)\right).$$

Theorem 2.3. *For the spectra $\sigma(T_1)$ of the PIO T_1 with a degenerate kernel* (3)*, the following formula holds:*

$$\sigma(T_1) = \{0\}\cup\left(\bigcup_{k=1}^{n} Essran(h_k)\right).$$

Proof. By proposition 2.2, we obtaine

$$\sigma(T_1) \subset \{0\}\cup\left(\bigcup_{k=1}^{n} Essran(h_k)\right).$$

However, by proposition 2.1 we have $0\in\sigma(T_1)$. Now we prove

$$\bigcup_{k=1}^{n} Essran(h_k) \subset \sigma(T_1).$$

Let $\lambda_0 \in Essran(h_{j_0})$, $\lambda_0 \neq 0$ and t_0 be arbitrary point from the subset $h_{j_0}^{-1}(\{\lambda_0\})$.

Put

$$V_i = \left\{t\in[c,d] : \frac{1}{i+1} < |t_0 - t| < \frac{1}{i}\right\},\quad i\in\mathbb{N}.$$

Then there exists $n_0\in\mathbb{N}$ such that $\mu_2(V_i)>0$ for all $i\geq n_0$. We consider the following sequence of orthonormal functions $\chi_p(y)\in L_2[c,d]$:

$$\chi_p(y) = \begin{cases} \frac{1}{\sqrt{\mu_2(V_p)}}, & y\in V_p, \\ 0, & y\overline{\in} V_p, \end{cases}$$

where $p\geq n_0$. We define by $f_p(x,y)\in L_2([a,b]\times[c,d])$ the orthonormal system of functions: $f_p(x,y) = \varphi_{j_0}(x)\chi_p(y)$, $p\geq n_0$. Then we have

$$(T_1-\lambda_0 E)f_p(x,y) = (h_{j_0}(y)-\lambda_0)f_p(x,y).$$

Hence,

$$\|(T_1-\lambda_0 E)f_p\| \leq \sqrt{esssup_{V_p}\left(h_{j_0}(y)-\lambda_0\right)^2},\quad p\geq n_0.$$

Since zero is an essential value of the function $h_{j_0}(y)-\lambda_0$, then for large $n_1\geq n_0$, there exists a small number δ_{n_1} such that

$$\mid h_{j_0}(y)-\lambda_0 \mid < \delta_{n_1} \text{ for almost all } y\in V_p,\ p\geq n_1.$$

Therefore

$$\| (T_1-\lambda_0 E)f_p \| < \delta_{n_1},\quad p\geq n_1,$$

i.e. $\lim\limits_{n\to\infty} \| (T_1-\lambda_0 E)f_n \|= 0$. This and the Weyl criterion for an essential spectrum of self-adjoint operators [25] imply $\lambda_0 \in \sigma_{ess}(T_1) \subset \sigma(T_1)$.

Proposition 2.4. *Any eigenvalue of the PIO T_1 is of infinite multiplicity.*

Proof. Let $\lambda \in \mathbb{R}\backslash\{0\}$ be an eigenvalue of T_1. Then there exists $f_0 \in L_2([a,b]\times [c,d])$, $\|f_0\| = 1$ such that $T_1 f_0 = \lambda f_0$. We define the measurable subset $\Omega_0 \subset [c,d]$:

$$\Omega_0 = \left\{ y \in [c,d] : \int_a^b |f_0(x,y)|^2 d\mu_1(x) \neq 0 \right\}.$$

Obviously, $\mu_2(\Omega_0) > 0$. Define the function:

$$f_0(x,y) = \begin{cases} \frac{f_0(x,y)}{\sqrt{\int_a^b |f_0(s,y)|^2 d\mu_1(s)}}, & x\in[a,b], y\in\Omega_0 \\ 0, & x\in[a,b], y\overline{\in}\Omega_0, \end{cases}$$

Then $f_0 \in L_2([a,b]\times[c,d])$, and $f_0 \neq 0$.

Let $\{\widetilde{\psi_k}\}_k \in \mathbb{N}$ be a system of orthonormal functions from $L_2(\Omega_0)$. Consider the sequence of functions from $L_2([a,b]\times[c,d])$:

$$f_k(x,y) = f_0(x,y)\psi_k(y), \ \ k\in\mathbb{N},$$

where

$$\psi_k(y) = \begin{cases} \widetilde{\psi_k}(y), & y\in\Omega_0 \\ 0, & y\overline{\in}\Omega_0. \end{cases}$$

Then

$$\int_a^b \int_c^d |f_n(x,y)|^2 \, d\mu_1(x) d\mu_2(y) = \int_{\Omega_0} \left|\widetilde{\psi_n}(y)\right|^2 d\mu_2(y) = 1,$$

and

$$(f_i,f_j) = \int_{\Omega_0} \widetilde{\psi}_i(y)\overline{\widetilde{\psi_j}(y)} d\mu_2(y) = 0$$

for $i\neq j$.

Clearly

$$T_1 f_k(x,y) = \lambda f_k(x,y), \ \ k\in\mathbb{N},$$

i.e. the number λ is an eigenvalue of the PIO T_1 of infinite multiplicity.

Corollary 2.5. *A discrete spectrum of the PIO T_1 with a degenerate kernel* (3) *is absent.*

Corollary 2.6. *If every function h_k, $k\in\{1,...,n\}$ is continuous and strictly monotone on $[c,d]$, then there is not an eigenvalue of the PIO T_1 different from zero.*

Consider the following projectors Q_j in the space $L_2([a,b]\times[c,d])$:

$$Q_j f(x,y) = \int_c^d \psi_j(y)\overline{\psi_j(t)} f(x,t) d\mu_2(t), \ \ j\in\{1,...,m\}.$$

Proposition 2.7. *If $\lambda \neq 0$ and $\lambda \overline{\in} \bigcup\limits_{j=1}^{m} Essran(p_j)$, then the operator $T_2 - \lambda E$ is invertible on $L_2([a,b]\times[c,d])$, and the operator $(T_2-\lambda E)^{-1}$ is bounded in $L_2([a,b]\times[c,d])$, moreover*

$$(T_2 - \lambda E)^{-1} f(x,y) = -\frac{1}{\lambda}\left(f(x,y) - \sum_{j=1}^{m} \frac{p_j(x)}{p_j(x)-\lambda} Q_j f(x,y)\right).$$

Theorem 2.8. *For the spectrum $\sigma(T_2)$ of the PIO T_2 with a degenerate kernel (4), the following formula is hold:*

$$\sigma(T_2) = \{0\} \cup \left(\bigcup_{j=1}^{m} Essran(p_j)\right).$$

3. Solvability of partial integral equations

We consider the Fredholm partial integral equation (PIE) of the second kind

$$f(x,y) - \tau(T_1+T_2)f(x,y) = g(x,y) \tag{6}$$

in the Hilbert space $L_2([a,b]\times[c,d])$, where f is an unknown function from $L_2([a,b]\times[c,d])$, $g \in L_2([a,b]\times[c,d])$ is a given function, and $\tau \in \mathbb{C}$ is the parameter of the equation.

The homogeneous PIE corresponding to (6) has the following form:

$$f(x,y) - \tau(T_1+T_2)f(x,y) = 0.$$

In this section, we reduce to some necessary results for PIE of the second kind.

Assume that $\tau \neq 0$ and $\tau^{-1} \in \rho(T_1)$. Then the operator $E - \tau T_1$ is invertible in $L_2([a,b]\times[c,d])$, and the operator $(E-\tau T_1)^{-1}$ is bounded on $L_2([a,b]\times[c,d])$, moreover, by proposition 2.2, we have

$$(E-\tau T_1)^{-1} f(x,y) = f(x,y) + \tau \sum_{k=1}^{n} \frac{h_k(y)}{1-\tau h_k(y)} P_k f(x,y).$$

Analogously, if $\tau \neq 0$ and $\tau^{-1} \in \rho(T_2)$, then the operator $E - \tau T_2$ is invertible in $L_2([a,b]\times[c,d])$, and the operator $(E-\tau T_2)^{-1}$ is bounded on $L_2([a,b]\times[c,d])$, moreover, by proposition 2.7,

$$(E-\tau T_2)^{-1} f(x,y) = f(x,y) + \tau \sum_{j=1}^{m} \frac{p_j(x)}{1-\tau p_j(x)} Q_j f(x,y).$$

Let $\tau \neq 0$ and $\tau^{-1} \in \rho(T_1)\cap\rho(T_2)$. We define compact operators $W_1(\tau)$ and $W_2(\tau)$ by the following formulas:

$$W_1(\tau) = (E-\tau T_2)^{-1} S_1(\tau) T_2, \quad W_2(\tau) = (E-\tau T_1)^{-1} S_2(\tau) T_1,$$

where

$$S_1(\tau) f(x,y) = \sum_{k=1}^{n} \frac{h_k(y)}{1-\tau h_k(y)} (P_k f)(x,y), \;\; S_2(\tau) f(x,y) = \sum_{i=1}^{m} \frac{p_i(x)}{1-\tau p_i(x)} (Q_i f)(x,y).$$

Lemma 3.1. *Let $\tau \neq 0$ and $\tau^{-1} \in \rho(T_1) \cap \rho(T_2)$. Then the following three homogenous Fredholm PIE of the second kind are equivalent:*

$$f - \tau(T_1 + T_2)f = 0, \tag{7}$$

$$f - \tau^2 W_1(\tau) f = 0, \tag{8}$$

$$f - \tau^2 W_2(\tau) f = 0. \tag{9}$$

In [19, 20, 21, 22, 23] lemmas similar to the lemma 3.1 for the case of general PIE's in $C([a, b] \times [c, d])$ with continuous kernels or kernels in $C(L_1)$ were proved. The scheme for the proof of lemma 3.1 can be seen from these works.

Assume that $\tau \neq 0$ and $\tau^{-1} \in \rho(T_1) \cap \rho(T_2)$. We denote by $\Delta_1(\tau)$ and $\Delta_2(\tau)$ the Fredholm determinants of the operators $E - \tau^2 W_1(\tau)$ and $E - \tau^2 W_2(\tau)$, respectively. Define in $\mathbb{C}$ the following subsets

$$\mathcal{R}_1 = \left\{\tau \in \mathbb{C} \setminus \{0\} : \tau^{-1} \in \rho(T_1) \cap \rho(T_2) \ \text{ and } \ \Delta_1(\tau) \neq 0\right\},$$

$$\mathcal{R}_2 = \left\{\tau \in \mathbb{C} \setminus \{0\} : \tau^{-1} \in \rho(T_1) \cap \rho(T_2) \ \text{ and } \ \Delta_2(\tau) \neq 0\right\},$$

$$\mathcal{D}_1 = \left\{\tau \in \mathbb{C} \setminus \{0\} : \tau^{-1} \in \rho(T_1) \cap \rho(T_2) \ \text{ and } \ \Delta_1(\tau) = 0\right\},$$

$$\mathcal{D}_2 = \left\{\tau \in \mathbb{C} \setminus \{0\} : \tau^{-1} \in \rho(T_1) \cap \rho(T_2) \ \text{ and } \ \Delta_2(\tau) = 0\right\}.$$

It follows from lemma 3.1 that $\mathcal{R}_1 = \mathcal{R}_2$ and $\mathcal{D}_1 = \mathcal{D}_2$. Put

$$\mathcal{R} = \mathcal{R}(T) = \mathcal{R}_1 \ \text{ and } \ \mathcal{D} = \mathcal{D}(T) = \mathcal{D}_1.$$

Then we obtain

$$\mathcal{R} \cup \mathcal{D} = \left\{\tau \in \mathbb{C} : \tau \in \mathbb{C} \setminus \{0\} \ \text{ and } \ \tau^{-1} \in \rho(T_1) \cap \rho(T_2)\right\} \tag{10}$$

and $\mathcal{R} \cap \mathcal{D} = \emptyset$.

Lemma 3.1 implies the following

Theorem 3.2. *Let $\tau \in \mathcal{R} \cup \mathcal{D}$. Homogenous PIE (7) has a non-trivial solution if and only if $\tau \in \mathcal{D}$.*

Theorem 3.3. *Let $\tau \in \mathcal{R}$. Then the Fredholm PIE of the second kind*

$$f - \tau(T_1 + T_2)f = g \tag{11}$$

has the unique solution $f_0 \in L_2([a, b] \times [c, d])$ for any $g \in L_2([a, b] \times [c, d])$.

4. Spectrum of the PIO $T_1 + T_2$

We put

$$\mathcal{D}_0 = \mathcal{D}_0(T) = \left\{\xi : \xi = \frac{1}{\tau}, \ \ \tau \in \mathcal{D}(T)\right\}.$$

Lemma 4.1. *For each $\lambda \in \mathbb{C} \setminus (\sigma(T_1) \cup \sigma(T_2) \cup \mathcal{D}_0(T))$, the resolvent $R_\lambda(T)$ of the PIO $T = T_1 + T_2$ exists and is bounded on $L_2([a, b] \times [c, d])$.*

Proof. Let $\lambda \in \mathbb{C} \setminus (\sigma(T_1) \cup \sigma(T_2) \cup \mathcal{D}_0(T))$. Then $\lambda \neq 0$, and $\lambda \in \rho(T_1) \cap \rho(T_2)$. Then operators $E - \frac{1}{\lambda}T_1$ and $E - \frac{1}{\lambda}T_2$ are injective and

$$\left(E - \frac{1}{\lambda}T_1\right)^{-1} = E + \frac{1}{\lambda}S_1\left(\frac{1}{\lambda}\right), \quad \left(E - \frac{1}{\lambda}T_2\right)^{-1} = E + \frac{1}{\lambda}S_2\left(\frac{1}{\lambda}\right).$$

However, from $\lambda \overline{\in} \mathcal{D}_0(T)$ and by (10), we obtain $\Delta_1\left(\frac{1}{\lambda}\right) \neq 0$. Consequently, the operator $E - \frac{1}{\lambda^2}W_1\left(\frac{1}{\lambda}\right)$ is injective. By the other hand, we have

$$T_1 + T_2 - \lambda E = (T_1 - \lambda E)(E + (T_1 - \lambda E)^{-1}T_2) =$$

$$= (T_1 - \lambda E)\left[E - \frac{1}{\lambda}\left(E + \frac{1}{\lambda}S_1\left(\frac{1}{\lambda}\right)\right)T_2\right] =$$

$$= (T_1 - \lambda E)\left(E - \frac{1}{\lambda}T_2\right)\left(E - \frac{1}{\lambda^2}\left(E - \frac{1}{\lambda}T_2\right)^{-1}S_1\left(\frac{1}{\lambda}\right)T_2\right) =$$

$$= -\lambda\left(E - \frac{1}{\lambda}T_1\right)\left(E - \frac{1}{\lambda}T_2\right)\left(E - \frac{1}{\lambda^2}W_1\left(\frac{1}{\lambda}\right)\right).$$

Hence,

$$R_\lambda(T) = -\frac{1}{\lambda}\left(E - \frac{1}{\lambda^2}W_1\left(\frac{1}{\lambda}\right)\right)^{-1}\left(E - \frac{1}{\lambda}T_2\right)^{-1}\left(E - \frac{1}{\lambda}T_1\right)^{-1}.$$

Boundedness of the operator $R_\lambda(T)$ follows from the last equality.

Corollary 4.2. *For the resolvent operator $R_\lambda(T)$ of the PIO $T = T_1 + T_2$, the formula*

$$R_\lambda(T) = -\frac{1}{\lambda}\left(E - \frac{1}{\lambda^2}W_1\left(\frac{1}{\lambda}\right)\right)^{-1}\left(E - \frac{1}{\lambda}T_2\right)^{-1}\left(E - \frac{1}{\lambda}T_1\right)^{-1}$$

holds for each $\lambda \in \mathbb{C} \setminus (\sigma(T_1) \cup \sigma(T_2) \cup \mathcal{D}_0(T))$.

Proposition 4.3. *Zero is an eigenvalue of infinite multiplicity of the PIO $T = T_1 + T_2$.*

Proof. Define by $\mathcal{L}$ the subspace of a Hilbert space $L_2([a,b] \times [c,d])$ constructed by the orthogonal system $\{\varphi_i(x)\psi_j(y)\}_{i=\overline{1,n},\ j=\overline{1,m}}$. Then $\dim \mathcal{L} = m \times n$ and for subspace $\mathcal{H}_0 = L_2([a,b] \times [c,d]) \ominus \mathcal{L}$, we have $\dim \mathcal{H}_0 = \infty$.

Let $\{f_k\}_{k\in\mathbb{N}}$ be an orthonormal basis in $\mathcal{H}_0$. It is obvious that

$$T_1 f_p(x,y) = \sum_{i=1}^{n} \int_a^b \varphi_i(x)\overline{\varphi_i(s)}h_i(y)f_k(s,y)d\mu_1(s) = 0, \ \ p \in \mathbb{N},$$

$$T_2 f_p(x,y) = \sum_{j=1}^{m} \int_c^d p_j(x)\psi_j(y)\overline{\psi_j(t)}f_k(x,t)d\mu_2(t) = 0, \ \ p \in \mathbb{N}.$$

i.e.

$$(T_1 + T_2)f_k(x,y) = 0, \ \ k \in \mathbb{N}.$$

Proposition 4.4. *The inclusion $\sigma(T_1) \cup \sigma(T_2) \subset \sigma_{ess}(T_1 + T_2)$ holds.*

Proof. We show that $\sigma(T_1) \subset \sigma_{ess}(T_1+T_2)$ (the inclusion $\sigma(T_2) \subset \sigma_{ess}(T_1+T_2)$ can be proved analogously). We have

$$\sigma(T_1) = \{0\} \cup \left(\bigcup_{i=1}^{n} Essran(h_i) \right).$$

By proposition 4.3, $0 \in \sigma_{ess}(T_1+T_2)$. Assume that $\lambda_0 \in Essran(h_{i_0})$, $\lambda_0 \neq 0$, where $i_0 \in \{1, 2, ..., n\}$ and y_0 is arbitrary point of the set $h_{i_0}^{-1}(\{\lambda_0\})$. Put

$$V_k = \left\{ y \in [c,d] : \frac{1}{k+1} < |y_0 - y| < \frac{1}{k} \right\}, \quad k \in \mathbb{N}.$$

Then there exists $k_0 \in \mathbb{N}$ such that $\mu_2(V_k) > 0$ for all $k \geq k_0$, and $\lim\limits_{n\to\infty} \mu_2(V_k) = 0$. Consider the sequence of orthonormal functions $\Phi_k(y) \in L_2[c,d]$:

$$\Phi_k(y) = \frac{\chi_{V_k}(y)}{\sqrt{\mu_2(V_k)}}, \quad k \in \mathbb{N}, \quad k \geq k_0.$$

Put

$$f_k(x,y) = \varphi_{i_0}(x)\Phi_k(y), \quad k \geq k_0.$$

Then the system of functions $\{f_k(x,y)\}_{k\geq k_0}$ of $L_2([a,b]\times[c,d])$ is an orthonormal system.
Now we prove that $\lim\limits_{k\to\infty} \|(T_1+T_2-\lambda_0 E)f_k\| = 0$. However, $\lim\limits_{k\to\infty} \|(T_1-\lambda_0 E)f_k\| = 0$ (see the proof of theorem 2.3). We show that $\lim\limits_{k\to\infty} \|T_2 f_k\| = 0$.

We define operators A_i, $i = 1, ..., m$ in the following way:

$$A_i f(x,y) = \int_c^d p_i(x)\psi_i(y)\overline{\psi_i(t)} f(x,t)\, d\mu_2(t), \quad f \in L_2([a,b]\times[c,d]).$$

Then

$$\|A_i f_k\|^2 \leq \int_a^b \int_c^d \left(\int_c^d |\, p_i(x)\,| \cdot |\, \psi_i(t)\,| \cdot |\, \psi_i(y)\,| \cdot |\, \varphi_{i_0}(x)\,| \cdot |\, \Phi_k(t)\,|\, d\mu_2(t) \right)^2 d\mu_1(x)\cdot$$

$$\cdot d\mu_2(y) \leq C_i^2 \left(\int_c^d |\, \psi_i(t)\,| \cdot \Phi_k(t) d\mu_2(t) \right)^2 = \frac{C_i^2}{\mu_2(V_k)} \left(\int_{V_k} |\, \psi_i(t)\,|\, d\mu_2(t) \right)^2 \leq$$

$$\leq C_i^2 \cdot \int_{V_k} |\, \psi_i(t)\,|^2\, d\mu(t), \quad i \in \{1, ..., n\},$$

where $C_i = esssup_{[a,b]} \mid p_i(x) \mid$. Since Lebesgue integrals are absolute continuous, we obtain

$$\lim_{k\to\infty} \int_{V_k} |\, \psi_i(t)\,|^2\, d\mu_2(t) = 0$$

at $\lim\limits_{k\to\infty} \mu_2(V_k) = 0$ and $\|\psi_i\| = 1$.

Thus, we get $\lim\limits_{k\to\infty} \|A_i f_k\| = 0$, $i \in \{1, ..., m\}$.

However,

$$\|T_2 f_k\| \leq \sum_{i=1}^{m} \|A_i f_k\|,$$

what follows $\lim\limits_{k\to\infty}\|T_2 f_k\| = 0$. Hence, $\lim\limits_{k\to\infty}\|(T_1+T_2-\lambda_0 E)f_k\| = 0$. Finally, by the Weyl criterion for an essential spectrum of self-adjoint operators [25], $\lambda_0 \in \sigma_{ess}(T_1+T_2)$.

Proposition 4.5. *Each $\lambda \in \mathcal{D}_0(T)$ is an eigenvalue of finite multiplicity of the PIO $T = T_1 + T_2$.*

Proof. Let $\lambda \in \mathcal{D}_0(T)$. Then $\lambda \neq 0$ and $\Delta_1\left(\frac{1}{\lambda}\right) = 0$, where $\Delta_1(\tau)$ is the Fredholm determinant of the operator $E - \tau^2 W_1(\tau)$. It means that the number 1 is an eigenvalue of the compact integral operator $\frac{1}{\lambda^2}W_1\left(\frac{1}{\lambda}\right)$. By lemma 3.1, the number λ is an eigenvalue of the PIO T_1+T_2. Since the following integral equations

$$f - \frac{1}{\lambda}(T_1+T_2)f = 0$$

and

$$f - \frac{1}{\lambda^2}W_1\left(\frac{1}{\lambda}\right)f = 0$$

are equivalent, the number λ is an eigenvalue of finite multiplicity of T_1+T_2 because of every eigenvalue $\alpha \neq 0$ of compact operators is of finite multiplicity.

The next theorem follows from lemma 4.1 and propositions 4.4, 4.5.

Theorem 4.6. *For the spectrum $\sigma(T)$ of the PIO $T = T_1 + T_2$ with a degenerate kernels, the following formula*

$$\sigma(T_1+T_2) = \{0\} \cup \left(\bigcup_{i=1}^{n} Essran(h_i)\right) \cup \left(\bigcup_{j=1}^{m} Essran(p_j)\right) \cup \mathcal{D}_0(T)$$

holds.

5. Discrete spectrum of the PIO $T_1 + T_2$

Put $G = \mathbb{C} \setminus (\sigma(T_1) \cup \sigma(T_2))$. It is well-known that spectra of a linear bounded self-adjoint operators are compact set in the set of all real numbers. Consequently, the set $\sigma(T_1) \cup \sigma(T_2)$ is a compact subset in $\mathbb{R}$. Therefore the set G is an open subset in $\mathbb{C}$ and G is unbounded domain in $\mathbb{C}$.

For each $\lambda \in G$, we consider the kernel of the compact integral operator $W_1\left(\frac{1}{\lambda}\right)$ given as follows:

$$W_1\left(\frac{1}{\lambda}\right) = \left(E - \frac{1}{\lambda}T_2\right)^{-1} S_1\left(\frac{1}{\lambda}\right)T_2.$$

Proposition 5.1. *For the kernel $\mathcal{K}(x,y;s,t|\lambda)$ $(\lambda \in G)$ of the Fredholm integral operator $W_1\left(\frac{1}{\lambda}\right)$, the equality*

$$\mathcal{K}(x,y;s,t|\lambda) = \lambda \sum_{j=1}^{n}\sum_{k=1}^{m} F_{k,j}(x,y;\lambda)B_{k,j}(s,t), \tag{12}$$

is valid, where

(13)

$$F_{k,j}(x,y;\lambda) = \varphi_j(x)\left(\frac{\psi_k(y)h_j(y)}{\lambda - h_j(y)} + \sum_{i=1}^{m}\frac{p_i(x)\psi_i(y)}{\lambda - p_i(x)}\int_c^d \frac{h_j(\xi)}{\lambda - h_j(\xi)}\psi_k(\xi)\overline{\psi_i(\xi)}d\mu_2(\xi)\right),$$

$$B_{k,j}(s,t) = p_k(s)\overline{\varphi_j(s)} \cdot \overline{\psi_k(t)}.$$

Proof. Let $\lambda \in G$. Then $\lambda \in \rho(T_1) \cap \rho(T_2)$, and we get

$$\left(E - \frac{1}{\lambda}T_2\right)^{-1} = E + \frac{1}{\lambda}S_2\left(\frac{1}{\lambda}\right).$$

However,

$$W_1\left(\frac{1}{\lambda}\right) = S_1\left(\frac{1}{\lambda}\right)T_2 + \frac{1}{\lambda}S_2\left(\frac{1}{\lambda}\right)S_1\left(\frac{1}{\lambda}\right)T_2.$$

For each $f \in L_2([a,b]\times[c,d])$, using representations of operators $S_1(\tau)$ and $S_2(\tau)$, we obtain

$$S_1\left(\frac{1}{\lambda}\right)T_2 f(x,y) = \sum_{j=1}^{n}\sum_{k=1}^{m}\int_a^b\int_c^d K_{k,j}(x,y;\lambda)B_{k,j}(s,t)f(s,t)d\mu_1(s)d\mu_2(t),$$

$$S_2\left(\frac{1}{\lambda}\right)S_1\left(\frac{1}{\lambda}\right)T_2 f(x,y) = \sum_{j=1}^{n}\sum_{k=1}^{m}\int_a^b\int_c^d G_{k,j}(x,y;\lambda)B_{k,j}(s,t)f(s,t)d\mu_1(s)d\mu_2(t),$$

where

$$K_{k,j}(x,y;\lambda) = \frac{\lambda\varphi_j(x)\psi_k(y)h_j(y)}{\lambda - h_j(y)},$$

$$G_{k,j}(x,y;\lambda) = \lambda^2\sum_{i=1}^{m}\frac{p_i(x)\psi_i(y)}{\lambda - p_i(x)}\int_c^d \frac{h_j(\xi)}{\lambda - h_j(\xi)}\psi_k(\xi)\overline{\psi_i(\xi)}d\mu_2(\xi).$$

Hence, we obtain equality (12) for the kernel $\mathcal{K}(x,y;s,t|\lambda)$ of the integral operator $W_1\left(\frac{1}{\lambda}\right)$.

Set

$$\Gamma_1 = \{1,2,...,m\}, \quad \Gamma_2 = \{1,2,...,n\} \quad \text{and} \quad \Gamma = \Gamma_1\times\Gamma_2.$$

We can introduce the relation of partial order in the set Γ by the following way: for elements $\omega = (k_1,j_1) \in \Gamma$ and $\omega' = (k_2,j_2) \in \Gamma$, we write $\omega \le \omega'$ if $k_1 < k_2$ or $k_1 = k_2$, $j_1 \le j_2$. As the set Γ is finite, the set Γ is linear complete ordered, i.e. for arbitrary $\omega,\omega' \in \Gamma$, we have $\omega \le \omega'$ or $\omega' \le \omega$. Thus, we can give elements of Γ in the increase order:

$$\Gamma = \{\omega_1,\omega_2,...,\omega_{m\cdot(n-1)},\omega_{m\cdot n}\},$$

moreover

$$\omega_1 = (1,1) < \omega_2 = (1,2) < ... < \omega_n = (1,n) < \omega_{n+1} = (2,1) < ... < \omega_{m\cdot n} = (m,n).$$

Let $\lambda \in G$ be fixed. For every $\omega = (k,j) \in \Gamma$, we define the function $F_\omega(x,y;\lambda)$ on $[a,b]\times[c,d]$ by the following formula:

$$F_\omega(x,y;\lambda) = F_{k,j}(x,y;\lambda).$$

Consider the homogenous Fredholm integral equation

$$f(x,y) - \frac{1}{\lambda^2}W_1\left(\frac{1}{\lambda}\right)f(x,y) = 0, \quad f \in L_2([a,b]\times[c,d]). \tag{14}$$

Set

$$\int_a^b \int_c^d F_{\omega_i}(x,y;\lambda) f(x,y) d\mu(x) d\mu(y) = A_{\omega_i}(\lambda), \ \ i \in \{1, ..., m \cdot n\}.$$

Then the homogenous equation (14) turns into the equation

$$f(x,y) = \frac{1}{\lambda} \sum_{i=1}^{m \cdot n} A_{\omega_i}(\lambda) F_{\omega_i}(x,y;\lambda).$$

Let

$$\int_a^b \int_c^d F_{\omega_i}(x,y;\lambda) B_{\omega_l}(x,y) d\mu_1(x) d\mu_2(y) = \Pi_{i,l}(\lambda), \ \ i,l \in \{1, ..., m \cdot n\},$$

where

$$B_{\omega_l}(x,y) = B_{k_l, j_l}(x,y), \quad \omega_l = (k_l, j_l).$$

Then we obtain a system of homogenous linear algebraic equations for unknown numbers $A_{\omega_i}(\lambda)$:

$$A_{\omega_i}(\lambda) - \frac{1}{\lambda} \sum_{l=1}^{m \cdot n} \Pi_{i,l}(\lambda) A_{\omega_l}(\lambda) = 0, \ \ i \in \{1, ..., m \cdot n\}. \tag{15}$$

Lemma 5.2. *Let* $\lambda \in G$. *The homogenous integral equation* (14) *has a nontrivial solution if and only if* $\Delta(\lambda) = 0$, *where*

$$\Delta(\lambda) = \begin{vmatrix} \Pi_{1,1}(\lambda) - \lambda & \Pi_{1,2}(\lambda) & \dots & \Pi_{1,m\cdot n}(\lambda) \\ \Pi_{2,1}(\lambda) & \Pi_{2,2}(\lambda) - \lambda & \dots & \Pi_{2,m\cdot n}(\lambda) \\ \dots & \dots & \dots & \dots \\ \Pi_{m\cdot n,1}(\lambda) & \Pi_{m\cdot n,2}(\lambda) & \dots & \Pi_{m\cdot n,m\cdot n}(\lambda) - \lambda \end{vmatrix}.$$

Proof. Let $\lambda \in G$. Then equivalence of the Fredholm integral equation of the second kind (14) and the system of linear algebraic homogeneous equations (15) is clear. The determinant $\widetilde{\Delta}(\lambda)$ of the system of equations (15) has the following form:

$$\widetilde{\Delta}(\lambda) = \begin{vmatrix} 1 - \frac{\Pi_{1,1}(\lambda)}{\lambda} & -\frac{\Pi_{1,2}(\lambda)}{\lambda} & \dots & -\frac{\Pi_{1,m\cdot n}(\lambda)}{\lambda} \\ -\frac{\Pi_{2,1}(\lambda)}{\lambda} & 1 - \frac{\Pi_{2,2}(\lambda)}{\lambda} & \dots & -\frac{\Pi_{2,m\cdot n}(\lambda)}{\lambda} \\ \dots & \dots & \dots & \dots \\ -\frac{\Pi_{m\cdot n,1}(\lambda)}{\lambda} & -\frac{\Pi_{m\cdot n,2}(\lambda)}{\lambda} & \dots & 1 - \frac{\Pi_{m\cdot n,m\cdot n}(\lambda)}{\lambda} \end{vmatrix},$$

and

$$\widetilde{\Delta}(\lambda) = \left(-\frac{1}{\lambda}\right)^{m \cdot n} \Delta(\lambda).$$

It is well-known, the system of linear homogeneous equations (15) has nontrivial solution if and only if $\widetilde{\Delta}(\lambda) = 0$, i.e. $\Delta(\lambda) = 0$. However, we obtain that the homogeneous Fredholm equation (14) has nontrivial solution if and only if $\Delta(\lambda) = 0$.

Lemma 5.3. *The function $\Delta(z)$ is holomorphic in the domain G.*

Proof. Let $\omega \in \Gamma$. It is known, the function $F_\omega(\lambda) = F_\omega(x, y; \lambda)$ is holomorphic by λ in the domain G for almost all $(x, y) \in [a, b] \times [c, d]$, and for every $\lambda \in G$ the integral

$$\int_a^b \int_c^d F_\omega(x, y; \lambda) B_{\omega'}(x, y) d\mu_1(x) d\mu_2(y), \quad \omega, \omega' \in \Gamma$$

exists and is finite. Then for every $\omega = (i, l) \in \Gamma$, the function $\Pi_{i,l}(z)$ is a holomorphic function in G. Consequently, the function $\Delta(z)$ is a sum of holomorphic functions $F_{\omega_1}(z), F_{\omega_2}(z), ..., F_{\omega_{m\cdot n}}(z)$, i.e. $\Delta(z)$ is holomorphic in G.

Remark 5.4. *An analogue of Lemma 5.3 can be proved for the general PIE.*

Theorem 5.5. *The discrete spectrum of the PIO $T = T_1 + T_2$ coincides with the set $\mathcal{D}_0(T)$.*

Proof. Lemmas 3.1 and 5.2 imply

$$\mathcal{D}_0(T) = \{\lambda \in G : \Delta(\lambda) = 0\}.$$

By proposition 4.4, we have

$$\sigma(T_1) \cup \sigma(T_2) \subset \sigma_{ess}(T_1 + T_2).$$

By theorem 4.6, we obtain

$$\sigma_{disc}(T) \subset \mathcal{D}_0(T).$$

Let $\lambda_0 \in \mathcal{D}_0(T)$ be arbitrary. Then by proposition 4.5, the number λ_0 is an eigenvalue of finite multiplicity of the operator $T_1 + T_2$. Since the function $\Delta(z)$ is holomorphic in the G, arbitrary point λ form the $\mathcal{D}_0(T)$ is isolate in $\mathcal{D}_0(T)$. Then the point λ_0 is isolate in the spectrum $\sigma(T_1) \cup \sigma(T_2) \cup \mathcal{D}_0(T)$ of the operator T since $(\sigma(T_1) \cup \sigma(T_2)) \cap \mathcal{D}_0(T) = \emptyset$. Thus, by definition of a discrete spectrum, we obtain $\lambda_0 \in \sigma_{disc}(T)$, i.e. $\mathcal{D}_0(T) \subset \sigma_{disc}(T)$.

Theorems 4.6 and 5.5 implies

Theorem 5.6.

$$\sigma_{ess}(T_1 + T_2) = \{0\} \cup \left(\bigcup_{i=1}^n Essran(h_i) \right) \cup \left(\bigcup_{j=1}^m Essran(p_j) \right).$$

Example 5.7. *Let $h_i(y) \equiv a_i \in \mathbb{R} \backslash \{0\}$, $i \in \{1, ..., n\}$ and $p_j(x) \equiv b_j \in \mathbb{R} \backslash \{0\}$, $j \in \{1, ..., m\}$ for the kernels $k_1(x, s, y)$ and $k_2(x, t, y)$ of PIO T_1 and T_2.*

Then by theorem 2.3 and 2.8, we obtain

$$\sigma(T_1) = \{0, a_1, ..., a_n\}, \quad \sigma(T_2) = \{0, b_1, ..., b_m\}.$$

By theorem 5.6, we have

$$\sigma_{ess}(T_1 + T_2) = \{0, a_1, ..., a_n, b_1, ..., b_m\}.$$

We obtain from (13):

$$F_{k,j}(x,y;\lambda) = \frac{\lambda a_j}{(a_j-\lambda)(b_k-\lambda)}\varphi_j(x)\psi_k(y), \quad \lambda \in \sigma_{ess}(T_1+T_2),$$

and

$$B_{k,j}(s,t) = b_k\overline{\varphi_j(s)}\cdot\overline{\psi_k(t)}.$$

Then the homogeneous Fredholm integral equation (14) becomes the following form (16)

$$f(x,y) - \sum_{j=1}^{n}\sum_{k=1}^{m}\frac{a_j b_k}{(a_j-\lambda)(b_k-\lambda)}\int_a^b\int_c^d \varphi_j(x)\psi_k(y)\overline{\varphi_j(s)}\cdot\overline{\psi_k(t)}d\mu_1(s)d\mu_2(t) = 0.$$

However, using the property of Fredholm integral equations with a degenerate kernel, we obtain

$$\frac{a_j b_k}{(a_j-\lambda)(b_k-\lambda)} = 1, \;\; j \in \{1,...,n\}, \;\; k \in \{1,...,m\}, \;\; \lambda\overline{\in}\sigma_{ess}(T_1+T_2).$$

It means that

$$a_j b_k = (a_j-\lambda)(b_k-\lambda), \;\; \lambda\overline{\in}\{0\}\cup\{a_i\}\cup\{b_l\},$$

i.e. the integral equation (16) has nontrivial solution if and only if

$$\lambda = a_j + b_k\overline{\in}\{0\}\cup\{a_i\}\cup\{b_l\}.$$

Set

$$\Lambda = \{\lambda : \lambda = a_j + b_k\overline{\in}\{0, a_1, ..., a_n, b_1, ..., b_m\}, \;\; j = \overline{1,n}, \;\; k = \overline{1,m}\}.$$

Therefore according to theorem 5.5, we obtain $\sigma_{disc}(T_1+T_2) = \Lambda$ and theorem 4.6 implies

$$\sigma(T_1+T_2) = \{\lambda : \lambda = a+b, \;\; a \in \sigma(T_1), \;\; b \in \sigma(T_2)\}. \tag{17}$$

It should be noted, the equality (17) was proved for the PIO T in $L_p, p \geq 1$ with more general kernels $k_1(x,s,y) = k_1(x,s)$, $k_2(x,t,y) = k_2(t,y)$ in the paper [18], [24].

Remark 5.8. *It is known, that the discrete spectrum $\sigma_{disc}(K)$ for each self-adjoint Fredholm integral operator K with a degenerate kernel is finite (since $\sigma_{disc}(K)$ is the set of all eigenvalues of K different from zero). The following question is arisen: Does this property hold for the PIO $T = T_1 + T_2$ with a degenerate kernels (3) and (4) This question is still an open problem.*

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Department of mechanics and mathematics. National University of Uzbekistan, Vuzgorodok, 100174, Tashkent, Uzbekistan.
E-mail address: yusup62@mail.ru

Department of energetics, Tashkent State Technical University, Vuzgorodok, 100174, Tashkent, Uzbekistan.
E-mail address: arzikulov79@mail.ru

Department of mechanics and mathematics. National University of Uzbekistan, Vuzgorodok, 100174, Tashkent, Uzbekistan.
E-mail address: haydarov_imc@mail.ru